THRESHOLD ANALYSIS OF THE NON-LOCAL DISCRETE SCHRÖDINGER OPERATOR WITH ONE-RANK PERTURBATION

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Abstract

The behaviour of the embedded eigenvalues and resonances is discussed at the lower threshold of the essential spectrum of non-local discrete Schrödinger operators with the Kroneker δ - potential with the mass $\mu \geq 0$. This operator is constructed by taking a strictly increasing C function of the standard discrete Laplacian instead of the original one. The dependence of the existence of resonances on this function and the lattice dimension are explicitly derived. We study the limits of eigenvalues as $\mu Z \ +\infty$ and $\mu] \ \ \mu_0$, where μ_0 is the value of μ which provides there existence of the threshold resonance.

Аннотация

Обсуждается поведение вложенных собственных значений и резонансов на нижнем пороге существенного спектра нелокальных дискретных операторов Шредингера с потенциалом Кронекера δ и массой $\mu \geq 0$. Этот оператор строится путем взятия строго возрастающей C-функции стандартного дискретного лапласиана вместо исходной. Зависимость существования резонансов от этой функции и размера решетки выводится в явном виде. Исследуем пределы собственных значений как $\mu Z \ +\infty$ и $\mu] \ \ \mu_0$, где μ_0 — значение μ, обеспечивающее существование порогового резонанса.

Keywords

Essential spectrum, threshold resonance, threshold eigenvalue, regular point.

Ключевые слова

существенный спектр, пороговый резонанс, пороговое собственное значение, регулярная точка.

Introduction

In the fields of quantum mechanics, mathematical physics, mathematical analysis and related fields spectral properties of Schrödinger operators, including lattice Schrödinger operators and their solids the applications in physics are significant. The spectral properties of discrete Schrödinger operators with the standard dispersion relation function (i.e., behaves as $e(p)=\sum_{j=1}^{d}(1-\cos p_j)$ have been extensively studied in recent years (see e.g. [1-9] and references therein) because of their applications in the theory of ultracold atoms in optical lattices [10,11]. In particular, it is well-known that the existence of the discrete spectrum is strongly connected to the threshold phenomenon [7, 8, 12, 13], which plays an role in the existence of the Efimov effect in threebody systems [14-16]: if any two-body subsystem in a three-body system has no bound state below its essential spectrum and at least two two-body subsystem has a zero-energy resonance, then the corresponding three-body system has infinitely many bound states whose energies accumulate at the lower edge of the three-body essential spectrum.

In the works [17, 18], were considered in the $d-$ dimensional lattice a family of the discrete Shrödinger operators depending on two parameters with a potential constructed via the delta function. The existence of eigenvalues, threshold eigenvalues and threshold resonances and their dependence on the parameters of the operator and dimension of the lattice was studied.

The fourth order elliptic operators in the space R^d in particular, the biharmonic operator, play also a central role in a wide class of physical models such as linear elasticity theory, rigidity problems and in stream function formulation of Stoke's flows (see e.g. [14, 19] and references therein).

A representation of eigenvalues and eigenfunctions, asymptotic formula of eigenvalues and some spectral properties for the pseudo-differential operator and fractional Schrödinger operators have been considered in [20, 21, 22, 23]. In [27], the authors introduced a class of generalized Schrödinger operators whose kinetic term is given by so called Bernstein functions of the Laplacian.

In this paper, we consider generalized discrete Schrödinger operators (i.e., non-local discrete Schrödinger operators) which include discrete bilaplacian operators [17] discrete fractional Schrödinger operators, and others whose counterparts on the continuos L^2-space are currently much studied [24, 25, 26, 28, 29]. We investigate the existence of eigenvalues as well as threshold resonance and bound states of the non-local discrete Schrödinger operator defined by (2).

Non-local Schrödinger operator

Let $\mathrm{T}^d=[-\pi,\pi)^d$ be the d-dimensional torus $(d=1,2,...)$, $L^2(\mathrm{T}^d)$ be the Hilbert space of L^2-functions on T^d and $\ell^2(\mathrm{Z}^d)$ be the Hilbert space of ℓ^2- functions on the $d-$ dimensional lattice Z^d.

Let Δ be the standart discrete Laplacian on $\ell^2(\mathrm{Z}^d)$ defined by

$$(\Delta\hat{f})(x)=\frac{1}{2}\sum_{|s|=1}(\hat{f}(x+s)-\hat{f}(x)), \quad \hat{f}\in\ell^2(\mathrm{Z}^d),$$

and $\hat{V}=\mu\delta_{x,0}$ be a potential defined by the Kroneker δ-function with mass $\mu(\mu\in\mathrm{R})$ concentrated on the origin $x=0$ in Z^d:

$$(\hat{V}\hat{f})(x)=\begin{cases}\mu\hat{f}(x), & if \quad x=0\\ 0, & if \quad x\neq 0\end{cases}, \quad \hat{f}\in\ell^2(\mathrm{Z}^d), \quad x\in\mathrm{Z}^d.$$

Then the discrete Schrödinger operator with the δ-potential has the form

$$\hat{h}=-\Delta-\mu\delta_{x,0}, \quad \mu\in\mathrm{R}.$$

In order to define a non-local version of $\hat{h}$, we use the Fourier transform $\mathcal{F} : \ell^2(\mathbb{Z}^d) \to L^2(\mathbb{T}^d)$ defined by

$$(\mathcal{F}\hat{f})(p) = \frac{1}{(2\pi)^{\frac{d}{2}}} \sum_{x \in \mathbb{Z}^d} \hat{f}(x) e^{-i(x,p)}, \quad \hat{f} \in \ell^2(\mathbb{Z}^d), \quad p \in \mathbb{T}^d,$$

whose inverse acts from $L^2(\mathbb{T}^d)$ to $\ell^2(\mathbb{Z}^d)$ as

$$(\mathcal{F}^{-1} f)(x) = \frac{1}{(2\pi)^{\frac{d}{2}}} \int_{\mathbb{T}^d} f(t) e^{i(x,t)} dt, \quad f \in L^2(\mathbb{T}^d), \quad x \in \mathbb{Z}^d.$$

Hence the discrete Laplacian $-\Delta$ is transformed into the multiplication operator as $h_0 = -\mathcal{F}\Delta\mathcal{F}^{-1}$:

$$(h_0 f)(p) = e(p) f(p), \quad f \in L^2(\mathbb{T}^d),$$

by the function

$$e(p) = \sum_{i=1}^{d} (1 - \cos p_i), \quad p \in \mathbb{T}^d,$$

and $\mu\hat{V}$ is transformed into the rank one integral operator $\mu V = \mathcal{F}\hat{V}\mathcal{F}^{-1}$:

$$(Vf)(p) = \frac{1}{(2\pi)^d} \int_{\mathbb{T}^d} f(q) dq, \quad f \in L^2(\mathbb{T}^d), \quad \mu \in \mathbb{R}. \tag{1}$$

In this paper, we use a non-local discrete Laplacian $\Psi(-\Delta)$ defined for a suitable function Ψ by applying Fourier transform. For a given strictly increasing continuous function $\Psi \in C(0,\infty)$, we define the non-local discrete Laplacian $\Psi(-\Delta)$ by

$$\hat{h}_0 = \Psi(-\Delta) = \mathcal{F}^{-1}\Psi\big(e(p)\big)\mathcal{F}.$$

The momentum representation of the non-local discrete Schrödinger operator acts in the space $L^2(\mathbb{T}^d)$ as

$$h_\mu = h_0 - \mu V, \tag{2}$$

where h_0 is a multiplication operator by the function $\Psi(e(\cdot))$:

$$(h_0 f)(p) = \Psi(e(p)) f(p), \quad f \in L^2(\mathbb{T}^d),$$

and V is defined by (1).

Essential spectrum

Since h_μ is selfadjoint and V is a rank one operator, according to the Weyl's theorem on stability of essential spectrum, the following relation holds

$$\sigma_{\text{ess}}(h_0 - \mu V) = \sigma(h_0), \quad i.e., \quad \sigma_{\text{ess}}(h_\mu) = [\Psi_{\min}, \Psi_{\max}]$$

where $\Psi_{\min} = \Psi(e(0))$ and $\Psi_{\max} = \Psi(e(2d))$.

For any $\mu \in \mathbb{R}$, we define the Fredholm determinant of the operator h_μ as a function of the variable $z \in \mathbb{C} \setminus [\Psi_{\min}; \Psi_{\max}]$ as follows

$$\Delta(\mu, z) = 1 - \frac{\mu}{(2\pi)^d} \int_{\mathbb{T}^d} \frac{dq}{\Psi(e(q)) - z}.$$

Lemma 1: *The number* $z \in \mathbb{C} \setminus [\Psi_{\min};\Psi_{\max}]$ *is an eigenvalue of operator* h_μ *if and only if*

$$\Delta(\mu, z) = 0.$$

Proof. Suppose that number, $z \in \mathbb{C} \setminus [\Psi_{\min};\Psi_{\max}]$ is eigenvalue of operator h_μ. Then

$$h_\mu f = zf \qquad (3)$$

i.e.,

$$\Psi(e(p)) f(p) - \frac{\mu}{(2\pi)^d} \int_{\mathrm{T}^d} f(q)dq = zf(p) \qquad (4)$$

equation has a non-trivial solution $f \in L^2(\mathrm{T}^d)$. It is clear to see the equation (3) has a non-trivial solution, if and only if

$$C_f \left(1 - \frac{\mu}{(2\pi)^d} \int_{\mathrm{T}^d} \frac{dq}{\Psi(e(p)) - z} \right) = 0 \quad (5)$$

has a non-trivial solution $C_f \in \mathbb{C}$, where solutions of (4) and (5) are related by the equalities,

$$C_f = \int_{\mathrm{T}^d} f(q)dq$$

and

$$f(p) = \frac{\mu}{(2\pi)^d} \frac{C_f}{\Psi(e(p)) - z}$$

The equation (5) has a non-trivial solution if and only if.

$$\Delta(\mu, z) = 1 - \frac{\mu}{(2\pi)^d} \int_{\mathrm{T}^d} \frac{dq}{\Psi(e(q)) - z} = 0.$$

Definition 1: *(Threshold eigenvalue and threshold resonance). Let the a measurable (non-trivial) function* f *in* T^d *be solution of the equation* $h_\mu f = \Psi(e(0)) f$

a) If $f \in L^2(\mathrm{T}^d)$ we say that the number $\Psi(e(0))$ is a lower threshold eigenvalue of the operator h_μ.

b) If $f \in L^1(\mathrm{T}^d) \setminus L^2(\mathrm{T}^d)$ we say that the number $\Psi(e(0))$ is a lower threshold resonance of the operator h_μ.

c) If $f \in L^\varepsilon(\mathrm{T}^d) \setminus L^1(\mathrm{T}^d)$ for any $\varepsilon (0 < \varepsilon < 1)$ we say that number $\Psi(e(0))$ is a lower super threshold resonance of operator h_μ.

d) If $h_\mu f = \Psi(e(0)) f$ equation has only trivial solution, the number $\Psi(e(0))$ is a regular point of operator h_μ.

In order to obtain the main results, we assume that the following condition :

Hypothesis 1. Let

$$C_1 |x - e(0)|^\alpha \le |\Psi(x) - \Psi(e(0))| \le C_2 |x - e(0)|^\alpha$$

for some $0 < \alpha \le 1,$ where

$$C_1 = \lim\inf_{x\to 0+} \frac{\Psi(x)-\Psi(0)}{|x|^{\alpha}}, \quad C_2 = \lim\sup_{x\to 0-} \frac{\Psi(x)-\Psi(0)}{|x|^{\alpha}},$$

and $0 < C_1 < C_2$.

We enter the following definition:

$$\mu_0 = (2\pi)^d \left(\int_{\mathrm{T}^d} \frac{dq}{\Psi(e(q))-\Psi(e(0))} \right)^{-1}. \qquad (6)$$

Lemma 2: *The following two statements are true:*

(a) If $\alpha \ge d/2$, *then* $\mu_0 = 0$.

(b) If $\alpha < d/2$, *then* $0 < \mu_0 < \infty$.

Proof. (a) According to Hypothesis 3 the following relation holds

$$\int_{\mathrm{T}^d} \frac{dq}{\Psi(e(q))-\Psi(e(0))} < \infty \Leftrightarrow C\int_{U_\gamma(0)} \frac{dq}{(|q|^2)^{\alpha}} < \infty$$

for some $\gamma > 0$, where $U_\gamma(0) = \{p \in \mathrm{T}^d : |p-0| < \gamma\}$ is the $\gamma(\gamma > 0)$ neighborhood of the origin $p = 0$. If we convert the variables in the last integral to a spherical coordinate system it is appropriate to have

$$\int_{\mathrm{T}^d} \frac{dq}{\Psi(e(q))-\Psi(e(0))} < \infty \Leftrightarrow C\int_0^{\delta} \frac{r^{d-1}}{r^{2\alpha}} dr < \infty,$$

for some $C > 0$. From $d \le 2\alpha$ it follows that the last integral is divergent that is $\mu_0 = 0$.

(b) On the other hand, if $d > 2\alpha$ then the integral $\int_{\mathrm{T}^d} \frac{dq}{\Psi(e(q))-\Psi(e(0))} < \infty$ convergente, and hence $\mu_0 > 0$.

Lemma 3: *The following two statements are true:*

(a) If $\mu > \mu_0 > 0$, *the operator* h_μ *has a unique (simple) eigenvalue* $z(\mu)$ *in the interval* $(-\infty, \Psi(e(0)))$.

(b) If $\mu_0 = 0$ *i.e.* $2\alpha \ge d$ *the operator* h_μ *has a simple eigenvalue* $z(\mu)$ *in the interval* $(-\infty, \Psi(e(0)))$ *for any* $\mu > 0$.

Proof. (a) It follows from the relation

$$\frac{\partial}{\partial z}\Delta(\mu, z) = -\frac{\mu}{(2\pi)^d} \int_{\mathrm{T}^d} \frac{dq}{(\Psi(e(q))-z)^2} < 0, \quad z < \Psi(e(0)),$$

that the function $\Delta(\mu, z)$ is strictly decreasing on $(-\infty; \Psi(e(0)))$. Then the relations

$$\lim_{z\to -\infty} \Delta(\mu, z) = 1 \qquad (7)$$

and

$$\Delta(\mu, \Psi(e(0))) = 1 - \frac{\mu}{(2\pi)^d} \int_{\mathrm{T}^d} \frac{dq}{\Psi(e(q))-\Psi(e(0))} = 1 - \frac{\mu}{\mu_0} < 0$$

imply that the function $\Delta(\mu, z)$ has a simple zero $z = z(\mu)$ in $(-\infty; \Psi(e(0)))$.

(b) Since

$$\lim_{z\to\Psi(e(0))-}\Delta(\mu,z)=-\infty$$

and (7) the function $\Delta(\mu,\cdot)$ changes sings in $(-\infty,\Psi(e(0))$, end hence the proof of (b) is analogy of the proof of (a).

Define

$$f_0(p)=\frac{1}{\Psi(e(p))-\Psi(e(0))},\quad p\in\mathrm{T}^d \tag{8}$$

(a) If $\alpha<d/4,$ then inclusion $f_0\in L^2(\mathrm{T}^d)$ holds.

There holds the following lemma.

Lemma 4: *The following three statements are true:*

(b) If $d/4\le\alpha<d/2$, then $f_0\in L^1(\mathrm{T}^d)/L^2(\mathrm{T}^d)$ holds.

(c) If $\alpha\ge d/2$, then for some $\varepsilon\quad(0<\varepsilon<1)$ the relation $f_0\in L^\varepsilon(\mathrm{T}^d)$ is true.

Proof. The proof of this lemma we obtain from the validity of following estimates:

(a) $$f_0\in L^2(\mathrm{T}^d)\quad\Leftrightarrow\quad\int_{\mathrm{T}^d}\frac{dq}{(\Psi(e(q))-\Psi(e(0)))^2}<\infty\quad\Leftrightarrow\quad\int_{U_\gamma(0)}\frac{dq}{((|q|^2)^\alpha)^2}<\infty$$

$$\Leftrightarrow C\int_0^\delta\frac{r^{d-1}}{r^{4\alpha}}dr<\infty,\quad\Leftrightarrow\quad d>4\alpha\quad\Leftrightarrow\quad\alpha<d/4.$$

(b) $$f_0\in L^1(\mathrm{T}^d)/L^2(\mathrm{T}^d)\quad\Leftrightarrow\quad\int_{\mathrm{T}^d}\frac{dq}{(\Psi(e(q))-\Psi(e(0)))}<\infty\quad\Leftrightarrow\quad\int_{U_\gamma(0)}\frac{dq}{(|q|^2)^\alpha}<\infty$$

$$\Leftrightarrow C\int_0^\delta\frac{r^{d-1}}{r^{2\alpha}}dr<\infty,\quad\Leftrightarrow\quad d>2\alpha\quad\Leftrightarrow\quad d/4\le\alpha<d/2.$$

(c) $$f_0\in L^\varepsilon(\mathrm{T}^d)\quad\Leftrightarrow\quad\int_{\mathrm{T}^d}\frac{dq}{(\Psi(e(q))-\Psi(e(0)))^\varepsilon}<\infty\quad\Leftrightarrow\quad\int_{U_\gamma(0)}\frac{dq}{((|q|^2)^\alpha)^\varepsilon}<\infty$$

$$\Leftrightarrow C\int_0^\delta\frac{r^{d-1}}{r^{2\varepsilon\alpha}}dr<\infty,\quad\Leftrightarrow\quad d/2>\alpha\varepsilon$$

would be appropriate for any $\varepsilon, 0<\varepsilon<d/2\alpha$.

a) If $\alpha<d/4$ the number $\Psi(e(0))$ is the threshold eigenvalue of the operator h_μ

b) If $d/4\le\alpha<d/2$ the number $\Psi(e(0))$ is the threshold resonance of the operator h_μ.

c) If $\alpha\ge d/2$ the number $\Psi(e(0))$ is the regular point of the operator h_μ.

Now we will formulate the main results of this paper.

Theorem 1: *The following three statements are true:*

Proof. It is easy to see that the equation

$$h_\mu g=\Psi(e(0))g$$

has a non-zero solution g when $\mu=\mu_0$ and the function g satisfies

$$g(p)=\mu f_0(p)$$

where f_0 is a function defined in (8)

It is easy to see that, Definition 3 and Lemma 3 imply the proof.

Theorem 2: *Let μ_0 be given by (6) and $z(\mu)$ be an eigenvalue in Lemma 3. The function $\mu:(\mu_0;\infty)\to z(\mu)$ is real-analytic strictly decreasing convex in $(\mu_0;+\infty)$ and satisfies*

$$\lim_{\mu\to\mu_0} z(\mu)=\Psi(e(0))$$

and

$$\lim_{\mu\to+\infty}\frac{z(\mu)}{\mu}=-(2\pi)^d. \qquad (9)$$

Proof. The relation

$$\frac{\partial}{\partial z}\Delta(\mu,z)=-\frac{\mu}{(2\pi)^d}\int_{T^d}\frac{dq}{(\Psi(e(q))-z)^2}<0,\quad z<\Psi(e(0)),$$

and the Implicit Function Theorem provides the function $\mu:(\mu_0;+\infty)\to z(\mu)$ is real-analytic.

Moreover, computing the derivatives of the implicit function $z(\mu)$ we find

$$z'(\mu)=-\frac{1}{\mu}\int_{T^d}\frac{dq}{\Psi(e(q))-z(\mu)}\left(\int_{T^d}\frac{dq}{(\Psi(e(q))-e(\mu))^2}\right)^{-1},\quad \mu\neq 0 \qquad (10)$$

thus using $\mu(\Psi(e(q))-z(\mu))>0$ we get $z'(\mu)<0$, i.e. $\Psi(e(\cdot))$ is strictly decreasing in $R\setminus\{0\}$. Differentiating (10) on more time we get

$$z''(\mu)=-\frac{2z'(\mu)}{\mu}\int_{T^d}\frac{dq}{(\Psi(e(q))-z(\mu))^3}\cdot\int_{T^d}\frac{dq}{\Psi(e(q))-z(\mu)}\cdot\left(\int_{T^d}\frac{dq}{(\Psi(e(q))-z(\mu))^2}\right)^{-2}$$

Therefore $z''(\mu)>0$ (i.e. $z(\cdot)$ is strictly concave) for any μ.

To prove (9) first we let $\mu\to+\infty$ in

$$1=\frac{\mu}{(2\pi)^d}\int_{T^d}\frac{dq}{\Psi(e(q))-z(\mu)} \qquad (11)$$

and find $\lim_{\mu\to+\infty} e(\mu)=-\infty$. In particular, if μ is sufficiently large

$$\left|\frac{\Psi(e(q))}{z(\mu)}\right|<\frac{1}{2}$$

and hence by (11) and the Dominated Convergence Theorem

$$\lim_{\mu\to+\infty}\frac{z(\mu)}{\mu}=-\lim_{\mu\to+\infty}\int_{T^d}\frac{dq}{1-\frac{\Psi(e(q))}{z(\mu)}}=-\int_{T^d}dq=-(2\pi)^d.$$

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